

Lecture 7

More on Constitutive Relations, Uniform Plane Wave

As mentioned before, constitutive relations are important for us to solve only the first two of four Maxwell's equations. Assuming that \mathbf{J} is known or zero, then the first two vector equations have four vector unknowns: \mathbf{E} , \mathbf{H} , \mathbf{D} , and \mathbf{B} , which cannot be determined by solving only two equations. The additional two equations come from the constitutive relations. Constitutive relations are useful because they allow us to incorporate material properties into the solutions of Maxwell's equations. The material properties can be frequency dispersive, anisotropic, bi-anisotropic, inhomogeneous, lossy, conductive, nonlinear as well as spatially dispersive. The use of phasors or frequency domain method will further simplify the characterization of different media. Hence, we will also study uniform plane wave in such media, including lossy conductive media.

7.1 More on Constitutive Relations

As have been said, Maxwell's equations are not solvable until the constitutive relations are included. Here, we will look into depth more into various kinds of constitutive relations. Now that we have learned phasor technique which is a powerful tool for frequency domain analysis, we can study a more general constitutive relationship compared to what we have seen earlier.

7.1.1 Isotropic Frequency Dispersive Media

First let us look at the simple linear constitutive relation previously discussed for dielectric media where [32], [33][p. 82], [47]

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad (7.1.1)$$

We have a simple model where

$$\mathbf{P} = \varepsilon_0 \chi_0 \mathbf{E} \quad (7.1.2)$$

where χ_0 is the electric susceptibility. When used in the generalized Ampere's law, \mathbf{P} , the polarization density, plays an important role for the flow of the displacement current through space. The polarization density is due to the presence of polar atoms or molecules that behave like little dipoles in the presence of an electric field. For instance, water, which is H_2O , is a polar molecule that becomes a small dipole when an electric field is applied.

We can think of displacement current flow as capacitive coupling between the dipoles yielding polarization current that flows through space. Namely, for a source-free medium,

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \quad (7.1.3)$$



Figure 7.1: As a series of dipoles line up end to end, one can see a current flowing through the line of dipoles as they oscillate back and forth in their polarity. This is similar to how displacement current flows through a series of capacitors.

For example, for a sinusoidal oscillating field, the dipoles will flip back and forth giving rise to flow of displacement current just as how time-harmonic electric current can flow through a capacitor as shown in Figure 7.1.

The linear relationship above can be generalized to that of a linear time-invariant system [53], or that at any given space point \mathbf{r} [36][p. 212], [47][p. 330].

$$\mathbf{P}(\mathbf{r}, t) = \varepsilon_0 \chi_e(\mathbf{r}, t) \circledast \mathbf{E}(\mathbf{r}, t) \quad (7.1.4)$$

where \circledast here implies a convolution. In the frequency domain or the Fourier space, the above linear relationship becomes

$$\mathbf{P}(\mathbf{r}, \omega) = \varepsilon_0 \chi_0(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega), \quad (7.1.5)$$

or

$$\mathbf{D}(\mathbf{r}, \omega) = \varepsilon_0 [1 + \chi_0(\mathbf{r}, \omega)] \mathbf{E}(\mathbf{r}, \omega) = \varepsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega) \quad (7.1.6)$$

where $\varepsilon(\mathbf{r}, \omega) = \varepsilon_0 [1 + \chi_0(\mathbf{r}, \omega)]$ at any point \mathbf{r} in space. There is a rich variety of ways at which $\chi_0(\omega)$ can manifest itself. Such a permittivity $\varepsilon(\mathbf{r}, \omega)$ is often called the effective permittivity. Such media where the effective permittivity is a function of frequency are termed dispersive media, or frequency dispersive media.

The above concept of simple relation between flux and field can be adapted for magnetic flux and field. By a quirk of history, the magnetic flux density \mathbf{B} is related to the magnetic field \mathbf{H} and magnetization \mathbf{M} as

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \quad (7.1.7)$$

Defining a magnetic susceptibility χ_m such that $\mathbf{M} = \chi_m \mathbf{H}$, one gets the relationship that

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} \quad (7.1.8)$$

which is analogous to the relationship between electric flux \mathbf{D} and electric field \mathbf{E} .

7.1.2 Anisotropic Media

For anisotropic media [33][p. 83]

$$\begin{aligned} \mathbf{D} &= \varepsilon_0 \mathbf{E} + \varepsilon_0 \bar{\chi}_0(\omega) \cdot \mathbf{E} \\ &= \varepsilon_0 [\bar{\mathbf{I}} + \bar{\chi}_0(\omega)] \cdot \mathbf{E} = \bar{\varepsilon}(\omega) \cdot \mathbf{E} \end{aligned} \quad (7.1.9)$$

In the above, $\bar{\varepsilon}$ is a 3×3 matrix also known as a tensor in electromagnetics. The above implies that \mathbf{D} and \mathbf{E} do not necessarily point in the same direction: the meaning of anisotropy. (A tensor is a special kind of matrix that is often associated with a physical notion like the relation between two physical fields, whereas a matrix is not.)

Previously, we have assumed that χ_0 to be frequency independent. This is not usually the case as all materials have χ_0 's that are frequency dependent. (This will become clear later.) Also, since $\bar{\varepsilon}(\omega)$ is frequency dependent, we should view it as a transfer function where the input is \mathbf{E} , and the output \mathbf{D} . This implies that in the time-domain, the above relation becomes a time-convolution relation as in (7.1.4).

Similarly for conductive media,

$$\mathbf{J} = \sigma \mathbf{E}, \quad (7.1.10)$$

This can be used in Maxwell's equations in the frequency domain to yield the definition of complex permittivity. Using the above in Ampere's law in the frequency domain, we have

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\varepsilon \mathbf{E}(\mathbf{r}) + \sigma \mathbf{E}(\mathbf{r}) = j\omega \underline{\varepsilon}(\omega) \mathbf{E}(\mathbf{r}) \quad (7.1.11)$$

where the complex permittivity $\underline{\varepsilon}(\omega) = \varepsilon - j\sigma/\omega$. Notice that Ampere's law in the frequency domain with complex permittivity in (7.1.11) is no more complicated than Ampere's law for nonconductive media. The algebra for complex numbers is no more difficult than the algebra for real numbers.¹ This is one of the strengths of phasor technique.

For anisotropic conductive media, one has

$$\mathbf{J}(\omega) = \bar{\sigma}(\omega) \cdot \mathbf{E}(\omega), \quad (7.1.12)$$

Here, again, due to the tensorial nature of the conductivity $\bar{\sigma}$, the electric current \mathbf{J} and electric field \mathbf{E} do not necessarily point in the same direction.

The above assumes a local or point-wise relationship between the input and the output of a linear system. This need not be so. In fact, the most general linear relationship between $\mathbf{P}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ is

$$\mathbf{P}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \iiint_{-\infty}^{\infty} \bar{\chi}(\mathbf{r} - \mathbf{r}', t - t') \cdot \mathbf{E}(\mathbf{r}', t') d\mathbf{r}' dt' \quad (7.1.13)$$

¹Computer scientists call two systems having the same algebraic structure homomorphic. We will use the term homomorphism to denote such, even though its precise mathematical meaning is quite abstract.

The above is a general convolutional relationship in both space and time. In the Fourier transform space, by taking Fourier transform in both space and time, the above becomes

$$\mathbf{P}(\mathbf{k}, \omega) = \bar{\chi}(\mathbf{k}, \omega) \cdot \mathbf{E}(\mathbf{k}, \omega) \quad (7.1.14)$$

where

$$\bar{\chi}(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} \bar{\chi}(\mathbf{r}, t) \exp(j\mathbf{k} \cdot \mathbf{r} - j\omega t) d\mathbf{r} dt \quad (7.1.15)$$

(The $d\mathbf{r}$ integral above is actually a three-fold integral with $d\mathbf{r} = dx dy dz$.) Such a medium is termed spatially dispersive as well as frequency dispersive [36][p. 6], [56]. In general²

$$\bar{\epsilon}(\mathbf{k}, \omega) = 1 + \bar{\chi}(\mathbf{k}, \omega) \quad (7.1.16)$$

where

$$\mathbf{D}(\mathbf{k}, \omega) = \bar{\epsilon}(\mathbf{k}, \omega) \cdot \mathbf{E}(\mathbf{k}, \omega) \quad (7.1.17)$$

The above can be extended to magnetic field and magnetic flux yielding

$$\mathbf{B}(\mathbf{k}, \omega) = \bar{\mu}(\mathbf{k}, \omega) \cdot \mathbf{H}(\mathbf{k}, \omega) \quad (7.1.18)$$

for a general spatial and frequency dispersive magnetic material. In optics, most materials are non-magnetic, and hence, $\mu = \mu_0$, whereas it is quite easy to make anisotropic magnetic materials in radio and microwave frequencies, such as ferrites.

7.1.3 Bi-anisotropic Media

In the previous section, the electric flux \mathbf{D} depends on the electric field \mathbf{E} and the magnetic flux \mathbf{B} , on the magnetic field \mathbf{H} . The concept of constitutive relations can be extended to where \mathbf{D} and \mathbf{B} depend on both \mathbf{E} and \mathbf{H} . In general, one can write

$$\mathbf{D} = \bar{\epsilon}(\omega) \cdot \mathbf{E} + \bar{\xi}(\omega) \cdot \mathbf{H} \quad (7.1.19)$$

$$\mathbf{B} = \bar{\zeta}(\omega) \cdot \mathbf{E} + \bar{\mu}(\omega) \cdot \mathbf{H} \quad (7.1.20)$$

A medium where the electric flux or the magnetic flux is dependent on both \mathbf{E} and \mathbf{H} is known as a bi-anisotropic medium [33][p. 81].

7.1.4 Inhomogeneous Media

If any of the $\bar{\epsilon}$, $\bar{\xi}$, $\bar{\zeta}$, or $\bar{\mu}$ is a function of position \mathbf{r} , the medium is termed an inhomogeneous medium or a heterogeneous medium. There are usually no simple solutions to problems associated with such media [36].

²In the following, to be precise, one should replace the 1 with an identity operator, but it is generally implied.

7.1.5 Uniaxial and Biaxial Media

Anisotropic optical materials are often encountered in optics. Examples of them are the biaxial and uniaxial media, and discussions of them are often found in optics books [57–59]. They are optical materials where the permittivity tensor can be written as

$$\bar{\epsilon} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \quad (7.1.21)$$

When $\epsilon_1 \neq \epsilon_2 \neq \epsilon_3$, the medium is known as a biaxial medium. But when $\epsilon_1 = \epsilon_2 \neq \epsilon_3$, then the medium is a uniaxial medium.

In the biaxial medium case, all three components of the electric field feel different permittivity constants. But in the uniaxial medium, the electric field in the xy plane feels the same permittivity constant, but the electric field in the z direction feels a different permittivity constant. As shall be shown later, different light polarization will propagate with different behaviors through such a medium.

7.1.6 Nonlinear Media

In the previous cases, we have assumed that $\bar{\chi}_0$ is independent of the field \mathbf{E} . The relationships between \mathbf{P} and \mathbf{E} can be written more generally as

$$\mathbf{P} = \epsilon_0 \bar{\chi}_0(\mathbf{E}) \cdot \mathbf{E} \quad (7.1.22)$$

where the relationship can appear in many different forms. For nonlinear media, the relationship can be nonlinear as indicated in the above. It can be easily shown that the principle of linear superposition does not hold for the above equation, a root test of linearity. Nonlinear permittivity effect is important in optics. Here, the wavelength is short, and a small change in the permittivity or refractive index can give rise to cumulative phase delay as the wave has to propagate many wavelengths through a nonlinear optical medium [60–62]. Kerr optical nonlinearity, discovered in 1875, was one of the earliest nonlinear phenomena observed [33, 57, 60].

For magnetic materials, nonlinearity can occur in the effective permeability of the medium. In other words,

$$\mathbf{B} = \bar{\mu}(\mathbf{H}) \cdot \mathbf{H} \quad (7.1.23)$$

This nonlinearity is important even at low frequencies, and in electric machinery designs [63, 64], and magnetic resonance imaging systems [65]. The large permeability in magnetic materials is usually due to the formation of magnetic domains which can only happen at low frequencies. The \mathbf{B} - \mathbf{H} relation in an electric machinery is shown in Figure 7.2. The loss of the system is related to the area of the hysteresis loop.

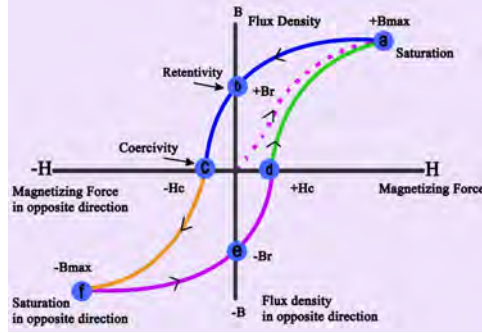


Figure 7.2: In an electric machinery, the relation between the \mathbf{B} flux and the magnetic \mathbf{H} is usually nonlinear, as shown in the picture. Moreover, the nonlinear system is not time-reversible, yielding a \mathbf{B} - \mathbf{H} relation as shown (courtesy of electricalacademia.com).

7.2 Wave Phenomenon in the Frequency Domain

We have seen the emergence of wave phenomenon in the time domain. Given the simplicity of the frequency domain method, it will be interesting to ask how this phenomenon presents itself for time-harmonic field. In the frequency domain, the source-free Maxwell's equations are [33][p. 429], [66][p. 107]

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu\mathbf{H}(\mathbf{r}) \quad (7.2.1)$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\varepsilon\mathbf{E}(\mathbf{r}) \quad (7.2.2)$$

Taking the curl of (7.2.1) and then substituting (7.2.2) into its right-hand side, one obtains

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu\nabla \times \mathbf{H}(\mathbf{r}) = \omega^2\mu\varepsilon\mathbf{E}(\mathbf{r}) \quad (7.2.3)$$

Again, using the identity that

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \cdot \nabla \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad (7.2.4)$$

and that $\nabla \cdot \mathbf{E} = 0$ in a source-free medium, (7.2.3) becomes

$$(\nabla^2 + \omega^2\mu\varepsilon)\mathbf{E}(\mathbf{r}) = 0 \quad (7.2.5)$$

This is known as the Helmholtz wave equation or just the Helmholtz equation.³

For lucidity of seeing the wave phenomenon, we let $\mathbf{E} = \hat{x}E_x(z)$, a field pointing in the x direction, but varying only in the z direction. Evidently, $\nabla \cdot \mathbf{E}(\mathbf{r}) = \partial E_x(z)/\partial x = 0$. Then with $\partial/\partial x = 0$ and $\partial/\partial y = 0$, (7.2.5) simplifies to

$$\left(\frac{d^2}{dz^2} + k^2\right)E_x(z) = 0 \quad (7.2.6)$$

³For a comprehensive review of this topic, one may read the lecture notes [45].

where $k^2 = \omega^2 \mu \epsilon = \omega^2 / c^2$ where $c = 1/\sqrt{\mu \epsilon}$ is the velocity of light. The general solution to (7.2.6) is of the form

$$E_x(z) = E_{0+} e^{-jkz} + E_{0-} e^{jkz} \quad (7.2.7)$$

One can convert the above back to the time domain using phasor technique, or by using that $E_x(z, t) = \Re e[E_x(z, \omega) e^{j\omega t}]$, yielding

$$E_x(z, t) = |E_{0+}| \cos(\omega t - kz + \alpha_+) + |E_{0-}| \cos(\omega t + kz + \alpha_-) \quad (7.2.8)$$

where we have assumed that $E_{0\pm}$ are complex numbers such that

$$E_{0\pm} = |E_{0\pm}| e^{j\alpha_{\pm}} \quad (7.2.9)$$

The physical picture of the above expressions can be appreciated by rewriting

$$\cos(\omega t \mp kz + \alpha_{\pm}) = \cos\left[\frac{\omega}{c}(ct \mp z) + \alpha_{\pm}\right] \quad (7.2.10)$$

where we have used the fact that $k = \frac{\omega}{c}$. The above functions are of the form $F(ct \mp z)$. As mentioned before in (3.2.14) and (3.2.15), these are traveling waves. One can see that the first term on the right-hand side of (7.2.8) is a sinusoidal plane wave traveling to the right, while the second term is a sinusoidal plane wave traveling to the left, both with velocity c . The above plane wave is uniform and a constant in the xy plane and propagating in the z direction. Hence, it is also called a uniform plane wave in 1D.

Moreover, for a fixed t or at $t = 0$, the sinusoidal functions are proportional to $\cos(\mp kz + \alpha_{\pm})$. This is a periodic function in z with period $2\pi/k$ which is the wavelength λ , or that

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \frac{2\pi f}{c} \quad (7.2.11)$$

One can see that because c is a humongous number in free space electromagnetics, λ can be very large. You can plug in the frequency of your local AM 920 station, operating at 920 KHz, to see that λ is approximately 320 m, the size of several football fields.

The above analysis still holds true even if ϵ and μ are dispersive, but are real numbers. In this case, the velocity c of the wave is the velocity of its phase, or the phase velocity of the mono-chromatic, time-harmonic, or CW wave.

7.3 Uniform Plane Waves in 3D

By repeating the previous derivation for a homogeneous, lossless, dispersive medium, the vector Helmholtz equation for a source-free medium is given by [45]

$$\nabla \times \nabla \times \mathbf{E} - \omega^2 \mu \epsilon \mathbf{E} = 0 \quad (7.3.1)$$

By the same derivation as before for the free-space case, since $\nabla \cdot \mathbf{E} = 0$ due to source-free medium, one has

$$\nabla^2 \mathbf{E} + \omega^2 \mu \epsilon \mathbf{E} = 0 \quad (7.3.2)$$

The general solution to (7.3.2) is hence

$$\mathbf{E} = \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z} = \mathbf{E}_0 e^{-j\mathbf{k} \cdot \mathbf{r}} \quad (7.3.3)$$

where $\mathbf{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$, $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$ and \mathbf{E}_0 is a constant vector. And upon substituting (7.3.3) into (7.3.2), it is seen that

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \epsilon = \mathbf{k} \cdot \mathbf{k} \quad (7.3.4)$$

This is called the dispersion relation for a plane wave. The above is also the equation for a sphere in a 3D \mathbf{k} space, which is also called the Ewald sphere.

In general, k_x , k_y , and k_z can be arbitrary and even complex numbers as long as this relation is satisfied. To simplify the discussion, we will focus on the case where k_x , k_y , and k_z are all real numbers. When this is the case, the vector function in (7.3.3) represents a uniform plane wave propagating in the \mathbf{k} direction. As can be seen, when $\mathbf{k} \cdot \mathbf{r} = \text{constant}$, it is represented by all points of \mathbf{r} that represents a flat plane (see Figure 7.3). This flat plane represents the constant phase wave front. By increasing the constant, we obtain different planes for progressively changing phase fronts.⁴

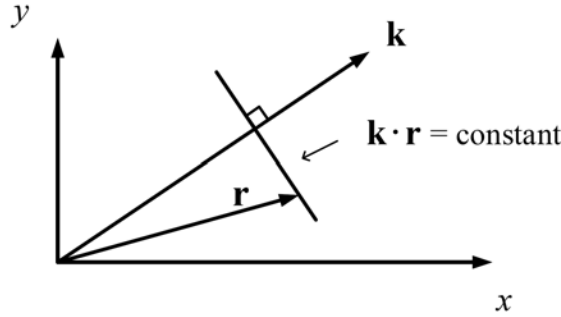


Figure 7.3: A figure showing the geometrical meaning of $\mathbf{k} \cdot \mathbf{r}$ equal to a constant. It is a flat plane that defines the wavefront of a plane wave.

Further, since $\nabla \cdot \mathbf{E} = 0$, we have then

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \nabla \cdot \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z} = \nabla \cdot \mathbf{E}_0 e^{-j\mathbf{k} \cdot \mathbf{r}} \\ &= (-\hat{x}jk_x - \hat{y}jk_y - \hat{z}jk_z) \cdot \mathbf{E}_0 e^{-j\mathbf{k} \cdot \mathbf{r}} \\ &= -j(\hat{x}k_x + \hat{y}k_y + \hat{z}k_z) \cdot \mathbf{E}_0 = 0 \end{aligned} \quad (7.3.5)$$

or that

$$\mathbf{k} \cdot \mathbf{E}_0 = \mathbf{k} \cdot \mathbf{E} = 0 \quad (7.3.6)$$

⁴In the $\exp(j\omega t)$ time convention, this phase front is decreasing, whereas in the $\exp(-i\omega t)$ time convention, this phase front is increasing. The $\exp(j\omega t)$ time convention is often used in electrical engineering, while the $\exp(-i\omega t)$ time convention is used in optics and physics.

Thus, both \mathbf{E} and \mathbf{E}_0 are orthogonal to \mathbf{k} for a uniform plane wave.

The above exercise shows that whenever \mathbf{E} is a plane wave, and when the ∇ operator operates on such a vector function, one can do the simple substitution that $\nabla \rightarrow -j\mathbf{k}$. Hence, in a source-free homogeneous medium,

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \quad (7.3.7)$$

the above equation simplifies to

$$-j\mathbf{k} \times \mathbf{E} = -j\omega\mu\mathbf{H} \quad (7.3.8)$$

or that

$$\mathbf{H} = \frac{\mathbf{k} \times \mathbf{E}}{\omega\mu} \quad (7.3.9)$$

Similar to (7.3.3), we can define

$$\mathbf{H} = \mathbf{H}_0 e^{-jk_x x - jk_y y - jk_z z} = \mathbf{H}_0 e^{-j\mathbf{k} \cdot \mathbf{r}} \quad (7.3.10)$$

Then using (7.3.3) in (7.3.9), it is clear that

$$\mathbf{H}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega\mu} \quad (7.3.11)$$

We can assume that \mathbf{E}_0 and \mathbf{H}_0 are real vectors. Then \mathbf{E}_0 , \mathbf{H}_0 and \mathbf{k} form a right-handed orthogonal system, or that $\mathbf{E}_0 \times \mathbf{H}_0$ points in the direction of \mathbf{k} . (This also implies that \mathbf{E} , \mathbf{H} and \mathbf{k} form a right-handed orthogonal system as well.) Such a wave, where the electric field and magnetic field are transverse to the direction of propagation, is called a transverse electromagnetic (TEM) wave. Figure 7.4 shows that $\mathbf{k} \cdot \mathbf{E} = 0$, and that $\mathbf{k} \times \mathbf{E}$ points in the direction of \mathbf{H} as shown in (7.3.9). Figure 7.4 also shows, as \mathbf{k} , \mathbf{E} , and \mathbf{H} are orthogonal to each other.

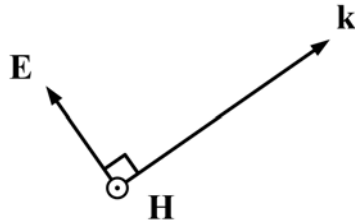


Figure 7.4: The \mathbf{E} , \mathbf{H} , and \mathbf{k} together form a right-hand coordinate system, obeying the right-hand rule. Namely, $\mathbf{E} \times \mathbf{H}$ points in the direction of \mathbf{k} .

Since in general, \mathbf{E}_0 and \mathbf{H}_0 can be complex vectors, because they are phasors, we need to show the more general case. From (7.3.9), one can show, using the “back-of-the-cab” formula,

assuming \mathbf{k} is real, that

$$\mathbf{E} \times \mathbf{H}^* = \mathbf{E} \cdot \mathbf{E}^* \frac{\mathbf{k}}{\omega\mu} = |\mathbf{E}|^2 \frac{\mathbf{k}}{\omega\mu} \quad (7.3.12)$$

(It is important to note that the magnitude square of a complex vector is $|\mathbf{E}|^2$ is $\mathbf{E} \cdot \mathbf{E}^*$, whereas that for a real vector, it is $\mathbf{E} \cdot \mathbf{E}$. The latter definition does not guarantee positive definiteness.) But $\mathbf{E} \times \mathbf{H}^*$ is the direction of power flow, and it is in fact in the \mathbf{k} direction. This is also required by the Poynting's theorem.

Furthermore, we can show in general that

$$|\mathbf{H}|^2 = \frac{|\mathbf{k} \times \mathbf{E}|^2}{(\omega\mu)^2} = \frac{\varepsilon}{\mu} |\mathbf{E}|^2 \quad (7.3.13)$$

or that

$$|\mathbf{H}| = \sqrt{\frac{\varepsilon}{\mu}} |\mathbf{E}| = \frac{1}{\eta} |\mathbf{E}| \quad (7.3.14)$$

where the quantity

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \quad (7.3.15)$$

is call the intrinsic impedance. For vacuum or free-space, it is about $377 \Omega \approx 120\pi \Omega$.

Notice that the above analysis holds true as long as ε and μ are real, but they can be frequency dispersive, since we are considering a mono-chromatic or time-harmonic field. Besides, for a mono-chromatic signal, the analysis in Section 7.2 still applies except that the velocity of light is now given by $c = 1/\sqrt{\mu\varepsilon}$. As we shall see, this velocity is the phase velocity of the mono-chromatic wave. In the above, when k_x , k_y , and k_z are not all real, the wave is known as an inhomogeneous wave.⁵

⁵The term inhomogeneous plane wave is used sometimes, but it is a misnomer since there is no more a planar wave front in this case.